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# CONTACT HYPERSURFACES IN KÄHLER MANIFOLDS

JÜRGEN BERNDT AND YOUNG JIN SUH

ABSTRACT. A contact hypersurface in a Kähler manifold is a real hypersurface for which the induced almost contact metric structure determines a contact structure. We carry out a systematic study of contact hypersurfaces in Kähler manifolds. We then apply these general results to obtain classifications of contact hypersurfaces with constant mean curvature in the complex quadric  $Q^n = SO_{n+2}/SO_n SO_2$  and its noncompact dual space  $Q^{n*} = SO_{n,2}^o/SO_n SO_2$  for  $n \geq 3$ .

## 1. INTRODUCTION

A contact manifold is a smooth  $(2n - 1)$ -dimensional manifold  $M$  together with a one-form  $\eta$  satisfying  $\eta \wedge (d\eta)^{n-1} \neq 0$ ,  $n \geq 2$ . The one-form  $\eta$  on a contact manifold is called a contact form. The kernel of  $\eta$  defines the so-called contact distribution  $\mathcal{C}$  on  $M$ . Note that if  $\eta$  is a contact form on a smooth manifold  $M$ , then  $\rho\eta$  is also a contact form on  $M$  for each smooth and everywhere nonzero function  $\rho$  on  $M$ . The origin of contact geometry can be traced back to Hamiltonian mechanics and geometric optics.

A standard example is a round sphere in an even-dimensional Euclidean space. Consider the sphere  $S^{2n-1}(r)$  with radius  $r \in \mathbb{R}_+$  in  $\mathbb{C}^n$  and denote by  $\langle \cdot, \cdot \rangle$  the inner product on  $\mathbb{C}^n$  given by  $\langle z, w \rangle = \operatorname{Re} \sum_{\nu=1}^n z_\nu \bar{w}_\nu$ . By defining  $\xi_z = -\frac{1}{r}iz$  for  $z \in S^{2n-1}(r)$  we obtain a unit tangent vector field  $\xi$  on  $S^{2n-1}(r)$ . We denote by  $\eta$  the dual one-form given by  $\eta(X) = \langle X, \xi \rangle$  and by  $\omega$  the Kähler form on  $\mathbb{C}^n$  given by  $\omega(X, Y) = \langle iX, Y \rangle$ . A straightforward calculation shows that  $d\eta(X, Y) = -\frac{2}{r}\omega(X, Y)$ . Since the Kähler form  $\omega$  has rank  $2(n - 1)$  on the kernel of  $\eta$  it follows that  $\eta \wedge (d\eta)^{n-1} \neq 0$ . Thus  $S^{2n-1}(r)$  is a contact manifold with contact form  $\eta$ . This argument for the sphere motivates a natural generalization to Kähler manifolds.

Let  $(\bar{M}, J, g)$  be a Kähler manifold of complex dimension  $n$  and let  $M$  be a connected oriented real hypersurface of  $\bar{M}$ . The Kähler structure on  $\bar{M}$  induces an almost contact metric structure  $(\phi, \xi, \eta, g)$  on  $M$ . The Riemannian metric on  $M$  is the one induced from the Riemannian metric on  $\bar{M}$ , both denoted by  $g$ . The orientation on  $M$  determines a unit normal vector field  $N$  of  $M$ . The so-called Reeb vector field  $\xi$  on  $M$  is defined by  $\xi = -JN$  and  $\eta$  is the dual one form on  $M$ , that is,  $\eta(X) = g(X, \xi)$ . The tensor field  $\phi$  on  $M$  is defined by  $\phi X = JX - \eta(X)N$ . Thus  $\phi X$  is the tangential component of  $JX$ . The tensor field  $\phi$  determines the fundamental 2-form  $\omega$  on  $M$  by  $\omega(X, Y) = g(\phi X, Y)$ .  $M$  is said to be a contact hypersurface if there exists an everywhere nonzero smooth function  $\rho$  on  $M$  such that  $d\eta = 2\rho\omega$ . It is clear that if  $d\eta = 2\rho\omega$  holds then  $\eta \wedge (d\eta)^{n-1} \neq 0$ , that is, every contact hypersurface in a Kähler manifold is a contact manifold.

Contact hypersurfaces in complex space forms of complex dimension  $n \geq 3$  have been investigated and classified by Okumura [6] (for the complex Euclidean space  $\mathbb{C}^n$  and the complex projective space  $\mathbb{C}P^n$ ) and Vernon [8] (for the complex hyperbolic space  $\mathbb{C}H^n$ ). In this paper we carry out a systematic study of contact hypersurfaces in Kähler manifolds. We will then apply our results to the complex quadric  $Q^n = SO_{n+2}/SO_nSO_2$  and its noncompact dual space  $Q^{n*} = SO_{n,2}^o/SO_nSO_2$ . Here we consider  $Q^n$  (resp.  $Q^{n*}$ ) equipped with the Kähler structure for which it becomes a Hermitian symmetric space with maximal (resp. minimal) sectional curvature 4 (resp.  $-4$ ). The classification results for these two spaces are as follows.

**Theorem 1.1.** *Let  $M$  be a connected orientable real hypersurface with constant mean curvature in the Hermitian symmetric space  $Q^n = SO_{n+2}/SO_nSO_2$  and  $n \geq 3$ . Then  $M$  is a contact hypersurface if and only if  $M$  is congruent to an open part of the tube of radius  $0 < r < \frac{\pi}{2\sqrt{2}}$  around the  $n$ -dimensional sphere  $S^n$  which is embedded in  $Q^n$  as a real form of  $Q^n$ .*

**Theorem 1.2.** *Let  $M$  be a connected orientable real hypersurface with constant mean curvature in the Hermitian symmetric space  $Q^{n*} = SO_{n,2}^o/SO_nSO_2$  and  $n \geq 3$ . Then  $M$  is a contact hypersurface if and only if  $M$  is congruent to an open part of one of the following contact hypersurfaces in  $Q^{n*}$ :*

- (i) *the tube of radius  $r \in \mathbb{R}_+$  around the Hermitian symmetric space  $Q^{(n-1)*}$  which is embedded in  $Q^{n*}$  as a totally geodesic complex hypersurface;*
- (ii) *a horosphere in  $Q^{n*}$  whose center at infinity is the equivalence class of an  $\mathcal{A}$ -principal geodesic in  $Q^{n*}$ ;*
- (iii) *the tube of radius  $r \in \mathbb{R}_+$  around the  $n$ -dimensional real hyperbolic space  $\mathbb{R}H^n$  which is embedded in  $Q^{n*}$  as a real form of  $Q^{n*}$ .*

The symbol  $\mathcal{A}$  refers to a circle bundle of real structures on  $Q^{n*}$  and the notion of  $\mathcal{A}$ -principal will be explained later. Every contact hypersurface in a Kähler manifold of constant holomorphic sectional curvature has constant mean curvature. Our results on contact hypersurfaces in Kähler manifolds suggest that it is natural to impose the condition of constant mean curvature in the more general setting.

In Section 2 we will develop the general theory of contact hypersurfaces in Kähler manifolds. In Section 3 we will apply these results to the complex quadric  $Q^n$  and its noncompact dual  $Q^{n*}$ .

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## 2. CONTACT HYPERSURFACES IN KÄHLER MANIFOLDS

Let  $\bar{M}$  be a Kähler manifold of complex dimension  $n$  and let  $M$  be a connected oriented real hypersurface of  $\bar{M}$ . The hypersurface  $M$  can be equipped with what is known as an almost contact metric structure  $(\phi, \xi, \eta, g)$  which consists of

1. a Riemannian metric  $g$  on  $M$  which is induced canonically from the Kähler metric (also denoted by  $g$ ) on  $\bar{M}$ ;
2. a tensor field  $\phi$  on  $M$  which is induced canonically from the complex structure  $J$  on  $\bar{M}$ : for all vectors fields  $X$  on  $M$  the vector field  $\phi X$  is obtained by projecting orthogonally the vector field  $JX$  onto the tangent bundle  $TM$ ;

3. a unit vector field  $\xi$  on  $M$  which is induced canonically from the orientation of  $M$ : if  $N$  is the unit normal vector field on  $M$  which determines the orientation of  $M$  then  $\xi = -JN$ ;
4. a one-form  $\eta$  which is defined as the dual of the vector field  $\xi$  with respect to the metric  $g$ , that is,  $\eta(X) = g(X, \xi)$  for all  $X \in TM$ .

The vector field  $\xi$  is also known as the Reeb vector field on  $M$ . The maximal complex subbundle  $\mathcal{C}$  of the tangent bundle  $TM$  of  $M$  is equal to  $\ker(\eta)$ .

Let  $S$  be the shape operator of  $M$  defined by  $SX = -\bar{\nabla}_X N$ , where  $\bar{\nabla}$  denotes the Levi Civita covariant derivative on  $\bar{M}$ . Applying  $J$  to both sides of this equation and using the fact that  $\bar{\nabla}J = 0$  on a Kähler manifold implies  $\phi SX = \nabla_X \xi$ , where  $\nabla$  is the induced Levi Civita covariant derivative on  $M$ . Using again  $\bar{\nabla}J = 0$  we get

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi. \quad (2.1)$$

Denote by  $\omega$  the fundamental 2-form on  $M$  given by  $\omega(X, Y) = g(\phi X, Y)$ .

**Proposition 2.1.** *The fundamental 2-form  $\omega$  on a real hypersurface in a Kähler manifold is closed, that is,  $d\omega = 0$ .*

*Proof.* By definition, we have

$$\begin{aligned} d\omega(X, Y, Z) &= d(\omega(Y, Z))(X) + d(\omega(Z, X))(Y) + d(\omega(X, Y))(Z) \\ &\quad - \omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y) \\ &= g(\nabla_X(\phi Y), Z) + g(\phi Y, \nabla_X Z) + g(\nabla_Y(\phi Z), X) + g(\phi Z, \nabla_Y X) \\ &\quad + g(\nabla_Z(\phi X), Y) + g(\phi X, \nabla_Z Y) \\ &\quad - g(\phi[X, Y], Z) - g(\phi[Y, Z], X) - g(\phi[Z, X], Y) \\ &= g((\nabla_X \phi)Y, Z) + g((\nabla_Y \phi)Z, X) + g((\nabla_Z \phi)X, Y). \end{aligned}$$

Inserting the expression for  $\nabla \phi$  as in (2.1) into the previous equation gives  $d\omega = 0$ .  $\square$

Motivated by the example  $S^{2n-1}(r) \subset \mathbb{C}^n$  we say that  $M$  is a contact hypersurface of  $\bar{M}$  if there exists an everywhere nonzero smooth function  $\rho$  on  $M$  such that  $d\eta = 2\rho\omega$  holds on  $M$  (Okumura [6]). It is clear that if this equation holds then  $\eta \wedge (d\eta)^{n-1} \neq 0$ , that is, every contact hypersurface of a Kähler manifold is a contact manifold. Note that the equation  $d\eta = 2\rho\omega$  means that  $d\eta(X, Y) = 2\rho g(\phi X, Y)$  for all tangent vector fields  $X, Y$  on  $M$ . Using the definition for the exterior derivative we obtain  $d\eta(X, Y) = d(\eta(Y))(X) - d(\eta(X))(Y) - \eta([X, Y]) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) = g(Y, \phi SX) - g(X, \phi SY) = g((S\phi + \phi S)X, Y)$ . Thus we have proved:

**Proposition 2.2.** *Let  $M$  be a connected orientable real hypersurface of a Kähler manifold  $\bar{M}$ . Then  $M$  is a contact hypersurface if and only if there exists an everywhere nonzero smooth function  $\rho$  on  $M$  such that*

$$S\phi + \phi S = 2\rho\phi. \quad (2.2)$$

A real hypersurface  $M$  of a Kähler manifold is called a Hopf hypersurface if the flow of the Reeb vector field is geodesic, that is, if every integral curve of  $\xi$  is a geodesic in  $M$ . This condition is equivalent to  $0 = \nabla_\xi \xi = \phi S\xi$ . Since the kernel of  $\phi$  is  $\mathbb{R}\xi$  this is equivalent to  $S\xi = \alpha\xi$  with the smooth function  $\alpha = g(S\xi, \xi)$ . Since  $\phi\xi = 0$ , equation (2.2) implies  $\phi S\xi = 0$  on a contact hypersurface, which shows that every contact hypersurface is a Hopf hypersurface. Let  $X \in \mathcal{C}$  be a principal curvature vector of  $M$  with

corresponding principal curvature  $\lambda$ . Then equation (2.2) implies  $S\phi X = (2\rho - \lambda)\phi X$ , that is,  $\phi X$  is a principal curvature vector of  $M$  with corresponding principal curvature  $2\rho - \lambda$ . Thus the mean curvature of  $M$  can be calculated from  $\alpha$  and  $\rho$  from the equation  $\text{tr}(S) = \alpha + 2(n-1)\rho$ . We summarize this in:

**Proposition 2.3.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}$ . Then the following statements hold:*

- (i)  *$M$  is a Hopf hypersurface, that is,  $S\xi = \alpha\xi$ .*
- (ii) *If  $X \in \mathcal{C}$  is a principal curvature vector of  $M$  with corresponding principal curvature  $\lambda$ , then  $JX = \phi X \in \mathcal{C}$  is a principal curvature vector of  $M$  with corresponding principal curvature  $2\rho - \lambda$ .*
- (iii) *The mean curvature of  $M$  is given by*

$$\text{tr}(S) = \alpha + 2(n-1)\rho. \quad (2.3)$$

For  $n = 2$  this gives a simple characterization of contact hypersurfaces:

**Proposition 2.4.** *Let  $M$  be a connected orientable real hypersurface of a 2-dimensional Kähler manifold  $\bar{M}^2$ . Then  $M$  is a contact hypersurface if and only if  $M$  is a Hopf hypersurface and  $\text{tr}(S) \neq \alpha$  everywhere.*

*Proof.* The "only if" part has been proved in Proposition 2.3. Assume that  $M$  is a Hopf hypersurface. Then we have  $S\xi = \alpha\xi$  and the maximal complex subbundle  $\mathcal{C}$  of  $TM$  is invariant under the shape operator  $S$  of  $M$ . Let  $X \in \mathcal{C}$  be a principal curvature vector with corresponding principal curvature  $\lambda$ . Since the rank of  $\mathcal{C}$  is equal to 2 the vector  $JX = \phi X$  must be a principal curvature vector of  $M$ . Denote by  $\mu$  the corresponding principal curvature. Then we have  $S\phi X + \phi SX = (\mu + \lambda)\phi X$  and  $S\phi(\phi X) + \phi S(\phi X) = -(\lambda + \mu)X = (\lambda + \mu)\phi(\phi X)$ . This shows that the equation  $S\phi + \phi S = 2\rho\phi$  holds with  $2\rho = (\lambda + \mu) = \text{tr}(S) - \alpha$ . It follows from Proposition 2.2 that  $M$  is a contact hypersurface precisely if  $\text{tr}(S) \neq \alpha$ .  $\square$

The previous result implies that there is a significant difference between the cases  $n = 2$  and  $n > 2$ . For example, using Proposition 2.4 we can construct many examples of locally inhomogeneous contact hypersurfaces in the complex projective plane  $\mathbb{C}P^2$ . In contrast, as was shown by Okumura ([6]), every contact hypersurface in the complex projective space  $\mathbb{C}P^n$  of dimension  $n > 2$  is an open part of a homogeneous hypersurface. Consider  $\mathbb{C}P^2$  being endowed with the standard Kähler metric of constant holomorphic sectional curvature 4. Let  $C$  be a complex curve in  $\mathbb{C}P^2$ . Then, at least locally and for small radii, the tubes around  $C$  are well-defined real hypersurfaces of  $\mathbb{C}P^2$ . All these real hypersurfaces are Hopf hypersurfaces with  $\alpha = 2\cot(2r)$  where  $r$  is the radius, and generically their mean curvature is different from  $2\cot(2r)$ . For this reason we focus here on the case  $n > 2$ .

**Proposition 2.5.** *Let  $M$  be a connected real hypersurface of an  $n$ -dimensional Kähler manifold  $\bar{M}^n$ ,  $n > 2$ , and assume that there exists an everywhere nonzero smooth function  $\rho$  on  $M$  such that  $d\eta = 2\rho\omega$ . Then  $\rho$  is constant.*

*Proof.* Taking the exterior derivative of the equation  $d\eta = 2\rho\omega$  and using the fact that  $\omega$  is closed gives  $0 = d^2\eta = 2d\rho \wedge \omega$ , or equivalently,

$$0 = d\rho(X)g(\phi Y, Z) + d\rho(Y)g(\phi Z, X) + d\rho(Z)g(\phi X, Y). \quad (2.4)$$

For  $X = \xi$ ,  $Y \in \mathcal{C}$  with  $\|Y\| = 1$ , and  $Z = \phi Y$ , this implies  $d\rho(\xi) = 0$ . Let  $X \in \mathcal{C}$ . Since  $n > 2$  we can choose a unit vector  $Y \in \mathcal{C}$  which is perpendicular to both  $X$  and  $\phi X$ . Inserting  $X, Y$  and  $Z = \phi Y$  into equation (2.4) gives  $d\rho(X) = 0$ . Altogether, since  $M$  is connected, this implies that  $\rho$  is constant.  $\square$

We denote by  $\bar{R}$  the Riemannian curvature tensor of  $\bar{M}$ . For  $p \in M$  and  $Z \in T_p \bar{M}$  we denote by  $Z_{\mathcal{C}}$  the orthogonal projection of  $Z$  onto  $\mathcal{C}$ .

**Proposition 2.6.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}$ . Then we have  $2(S^2 - 2\rho S + \alpha\rho)X = (\bar{R}(JN, N)JX)_{\mathcal{C}}$  for all  $X \in \mathcal{C}$ . In particular, for all  $X \in \mathcal{C}$  with  $SX = \lambda X$  we have  $2(\lambda^2 - 2\rho\lambda + \alpha\rho)X = (\bar{R}(JN, N)JX)_{\mathcal{C}}$ .*

*Proof.* Since  $M$  is a contact hypersurface, we know from Proposition 2.3 that  $S\xi = \alpha\xi$ . Using the Codazzi equation and Proposition 2.2 we get for arbitrary tangent vector fields  $X$  and  $Y$  that

$$\begin{aligned} g(\bar{R}(X, Y)\xi, N) &= g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y) \\ &= d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + 2g((S^2 - 2\rho S + \alpha\rho)\phi X, Y). \end{aligned}$$

For  $X = \xi$  this equation yields  $d\alpha(Y) = d\alpha(\xi)\eta(Y) + g(\bar{R}(Y, \xi)\xi, N)$ . Since  $\bar{M}$  is a Kähler manifold we have  $g(\bar{R}(Y, \xi)\xi, N) = g(\bar{R}(JY, J\xi)\xi, N) = g(\bar{R}(JY, N)\xi, N) = g(J\bar{R}(\xi, N)N, Y)$ , and therefore

$$d\alpha(Y) = d\alpha(\xi)\eta(Y) + g(J\bar{R}(\xi, N)N, Y). \quad (2.5)$$

Inserting this and the corresponding equation for  $d\alpha(X)$  into the previous equation gives  $0 = 2g((S^2 - 2\rho S + \alpha\rho)\phi X, Y) - g(\bar{R}(X, Y)\xi, N) - \eta(X)g(J\bar{R}(\xi, N)N, Y) + \eta(Y)g(J\bar{R}(\xi, N)N, X)$ . Choosing  $X \in \mathcal{C}$ , replacing  $X$  by  $\phi X$ , and using some standard curvature identities then leads to the equation in Proposition 2.6.  $\square$

**Proposition 2.7.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}$ . Then  $\alpha$  is constant if and only if  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere.*

*Proof.* First assume that  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere. From equation (2.5) we get  $d\alpha(Y) = d\alpha(\xi)\eta(Y)$  for all  $Y \in TM$ . Since  $\text{grad}^M \alpha = d\alpha(\xi)\xi$ , we can compute the Hessian  $\text{hess}^M \alpha$  by  $(\text{hess}^M \alpha)(X, Y) = g(\nabla_X \text{grad}^M \alpha, Y) = d(d\alpha(\xi))(X)\eta(Y) + d\alpha(\xi)g(\phi SX, Y)$ . As  $\text{hess}^M \alpha$  is a symmetric bilinear form, the previous equation implies  $0 = d\alpha(\xi)g((S\phi + \phi S)X, Y) = 2\rho d\alpha(\xi)g(\phi X, Y)$  for all vector fields  $X, Y$  on  $M$  which are tangential to  $\mathcal{C}$ . Since  $\rho$  is nonzero everywhere this implies  $d\alpha(\xi) = 0$  and hence  $\alpha$  is constant.

Conversely, assume that  $\alpha$  is constant. From (2.5) we get  $0 = g(J\bar{R}(\xi, N)N, Y) = g(\bar{R}(JN, N)N, JY)$  for all tangent vectors  $Y$  of  $M$ . Since  $J(TM) = \mathcal{C} \oplus \mathbb{R}N$  this implies that  $\bar{R}(JN, N)N \in \mathbb{R}JN$  everywhere.  $\square$

**Proposition 2.8.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}$ . Then we have*

$$\|S\|^2 = \text{tr}(S^2) = \alpha^2 + 2(n-1)\rho(2\rho - \alpha) - g(\overline{\text{Ric}}(N), N) + g(\bar{R}(JN, N)N, JN),$$

where  $\overline{\text{Ric}}$  is the Ricci tensor of  $\bar{M}$ .

*Proof.* We choose a local orthonormal frame field of  $\bar{M}$  along  $M$  of the form  $E_1, E_2 = JE_1, \dots, E_{2n-3}, E_{2n-2} = JE_{2n-3}, E_{2n-1} = \xi, E_{2n} = J\xi = N$ . Since  $\bar{M}$  is Kähler, its Ricci tensor  $\bar{Ric}$  can be calculated by

$$\bar{Ric}(X) = \sum_{\nu=1}^n \bar{R}(E_{2\nu-1}, JE_{2\nu-1})JX = \sum_{\nu=1}^n \bar{R}(E_{2\nu}, JE_{2\nu})JX$$

along  $M$  (see e.g. Proposition 4.57 in [1]). Using Propositions 2.6 and 2.3 we get

$$\begin{aligned} \text{tr}(S^2) &= \alpha^2 + \sum_{\nu=1}^{2n-2} g(S^2 E_\nu, E_\nu) \\ &= \alpha^2 + \sum_{\nu=1}^{2n-2} \left( 2\rho g(SE_\nu, E_\nu) - \alpha\rho g(E_\nu, E_\nu) + \frac{1}{2}g(\bar{R}(JN, N)JE_\nu, E_\nu) \right) \\ &= \alpha^2 + 2\rho(\text{tr}(S) - \alpha) - 2(n-1)\alpha\rho - \frac{1}{2} \sum_{\nu=1}^{2n-2} g(\bar{R}(E_\nu, JE_\nu)JN, N) \\ &= \alpha^2 + 4(n-1)\rho^2 - 2(n-1)\alpha\rho - (g(\bar{Ric}(N), N) - g(\bar{R}(N, JN)JN, N)) \\ &= \alpha^2 + 2(n-1)\rho(2\rho - \alpha) - g(\bar{Ric}(N), N) + g(\bar{R}(JN, N)N, JN), \end{aligned}$$

which proves the assertion.  $\square$

**Proposition 2.9.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}^n$ ,  $n > 2$ . Then we have  $d(\text{tr}(S))(X) = g(\bar{R}(JN, N)N, JX)$  for all  $X \in \mathcal{C}$ .*

*Proof.* Since  $M$  is a contact hypersurface, the equation  $\phi S + S\phi = 2\rho\phi$  holds, and since  $n > 2$ , the function  $\rho$  is constant. Differentiating this equation leads to

$$(\nabla_Y \phi)SX + \phi(\nabla_Y S)X + (\nabla_Y S)\phi X + S(\nabla_Y \phi)X = 2\rho(\nabla_Y \phi)X.$$

Using (2.1) this implies

$$0 = \eta(X)(S^2 Y + (\alpha - 2\rho)SY) - g(S^2 X + (\alpha - 2\rho)SX, Y)\xi + \phi(\nabla_Y S)X + (\nabla_Y S)\phi X.$$

We choose a local orthonormal frame field of  $M$  of the form  $E_1, E_2 = JE_1, \dots, E_{2n-3}, E_{2n-2} = JE_{2n-3}, E_{2n-1} = \xi$ . Contracting the previous equation with respect to this frame field and using the formulas for  $\text{tr}(S)$  and  $\text{tr}(S^2)$  according to Propositions 2.3 and 2.8, respectively, then gives

$$0 = (g(\bar{Ric}(N), N) - g(\bar{R}(JN, N)N, JN))\xi + \sum_{\nu=1}^{2n-1} \phi(\nabla_{E_\nu} S)E_\nu + \sum_{\nu=1}^{2n-2} (\nabla_{E_\nu} S)\phi E_\nu.$$

Using the Codazzi equation  $g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z) = g(\bar{R}(X, Y)Z, N)$  we get

$$\begin{aligned} \sum_{\nu=1}^{2n-1} g(\phi(\nabla_{E_\nu} S)E_\nu, X) &= - \sum_{\nu=1}^{2n-1} g((\nabla_{E_\nu} S)E_\nu, \phi X) = - \sum_{\nu=1}^{2n-1} g((\nabla_{E_\nu} S)\phi X, E_\nu) \\ &= - \sum_{\nu=1}^{2n-1} g((\nabla_{\phi X} S)E_\nu, E_\nu) - \sum_{\nu=1}^{2n-1} g(\bar{R}(E_\nu, \phi X)E_\nu, N) \\ &= -\text{tr}(\nabla_{\phi X} S) + g(\phi X, \bar{Ric}(N)) = -d(\text{tr}(S))(\phi X) + g(\phi X, \bar{Ric}(N)). \end{aligned}$$

Since  $\nabla_X S$  is symmetric and  $\phi$  is skewsymmetric, we get  $\sum_{\nu=1}^{2n-2} g((\nabla_X S)E_\nu, \phi E_\nu) = 0$ , and using again the Codazzi equation we obtain

$$\begin{aligned} \sum_{\nu=1}^{2n-2} g((\nabla_{E_\nu} S)\phi E_\nu, X) &= \sum_{\nu=1}^{2n-2} g((\nabla_{E_\nu} S)X, \phi E_\nu) = \sum_{\nu=1}^{2n-2} g(\bar{R}(E_\nu, X)\phi E_\nu, N) \\ &= \sum_{\nu=1}^{2n-2} g(\bar{R}(E_\nu, X)JE_\nu, N) = \sum_{\nu=1}^{2n-2} g(J\bar{R}(E_\nu, X)E_\nu, N) = \sum_{\nu=1}^{2n-2} g(\bar{R}(E_\nu, X)E_\nu, \xi) \\ &= -g(\bar{R}(JN, N)N, X) - g(X, \bar{Ric}(\xi)). \end{aligned}$$

Altogether this now implies

$$\begin{aligned} 0 &= (g(\bar{Ric}(N), N) - g(\bar{R}(JN, N)N, JN)) \eta(X) \\ &\quad - d(\text{tr}(S))(\phi X) + g(\phi X, \bar{Ric}(N)) - g(\bar{R}(JN, N)N, X) - g(X, \bar{Ric}(\xi)). \end{aligned}$$

Using the fact that the Ricci tensor  $\bar{Ric}$  and the complex structure  $J$  of a Kähler manifold commute one can easily see that  $\eta(X)g(\bar{Ric}(N), N) + g(\phi X, \bar{Ric}(N)) - g(X, \bar{Ric}(\xi)) = 0$ , and therefore  $d(\text{tr}(S))(\phi X) = -g(\bar{R}(JN, N)N, X) = -g((\bar{R}(JN, N)N)_C, X)$ . Replacing  $X$  by  $\phi X = JX$  for  $X \in \mathcal{C}$  then leads to the assertion.  $\square$

**Proposition 2.10.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}^n$ ,  $n > 2$ . Then  $M$  has constant mean curvature if and only if  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere.*

*Proof.* We first assume that  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N$  everywhere. We put  $f = \text{tr}(S)$  and  $\sigma = df(\xi)$ . From Proposition 2.9 we see that  $df = \sigma\eta$  and hence  $0 = d\sigma \wedge \eta + \sigma d\eta = d\sigma \wedge \eta + 2\rho\sigma\omega$ . In other words, we have  $0 = d\sigma(X)\eta(Y) - d\sigma(Y)\eta(X) + 2\rho\sigma g(\phi X, Y)$ . Replacing  $Y$  by  $\phi Y$  leads to  $0 = -d\sigma(\phi Y)\eta(X) + 2\rho\sigma g(\phi X, \phi Y)$ . By contracting this equation we obtain  $0 = 4(n-1)\rho\sigma$ . As  $\rho$  is nonzero everywhere this implies  $\sigma = 0$  and hence  $df = 0$ , which means that  $f = \text{tr}(S)$  is constant.

Conversely, assume that the mean curvature of  $M$  is constant. From Proposition 2.9 we get  $0 = g(\bar{R}(JN, N)N, JX)$  for all  $X \in \mathcal{C}$ . This implies that  $\bar{R}(JN, N)N$  is perpendicular to  $\mathcal{C}$  everywhere. Since  $g(\bar{R}(JN, N)N, N) = 0$  we conclude that  $\bar{R}(JN, N)N \in \mathbb{R}JN$ , that is,  $JN$  is an eigenvector of  $\bar{R}_N$  everywhere.  $\square$

From Proposition 2.7 and Proposition 2.10 we immediately get:

**Proposition 2.11.** *Let  $M$  be a contact hypersurface of a Kähler manifold  $\bar{M}^n$ ,  $n > 2$ . Then the following statements are equivalent:*

- (i)  $\alpha$  is constant;
- (ii)  $M$  has constant mean curvature;
- (iii)  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere.

The Riemannian universal covering of an  $n$ -dimensional Kähler manifold  $\bar{M}$  with constant holomorphic sectional curvature  $c$  is either the complex projective space  $\mathbb{CP}^n$ , the complex Euclidean space  $\mathbb{C}^n$  or the complex hyperbolic space  $\mathbb{CH}^n$  equipped with the standard Kähler metric of constant holomorphic sectional curvature  $c > 0$ ,  $c = 0$  and  $c < 0$ , respectively. The Riemannian curvature tensor  $\bar{R}$  of an  $n$ -dimensional Kähler manifold  $\bar{M}$  with constant holomorphic sectional curvature  $c$  is given by

$$\bar{R}(X, Y)Z = \frac{c}{4}(g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ).$$



This implies  $\bar{R}_N JN = \bar{R}(JN, N)N = cJN$ , and hence  $JN$  is an eigenvector of the Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere. We thus get from Proposition 2.11:

**Proposition 2.12.** *Let  $M$  be a contact hypersurface of an  $n$ -dimensional Kähler manifold  $\bar{M}$  with constant holomorphic sectional curvature,  $n \geq 3$ . Then  $M$  has constant mean curvature.*

For  $n \geq 3$ , the contact hypersurfaces in  $\mathbb{C}^n$  and  $\mathbb{C}P^n$  were classified by Okumura ([6]) and in  $\mathbb{C}H^n$  by Vernon ([8]). A remarkable consequence of their classifications is the following:

**Corollary 2.13.** *Every complete contact hypersurface in a simply connected complete Kähler manifold with constant holomorphic sectional curvature is homogeneous..*

For  $n = 2$ , however, there are more contact hypersurfaces, as we will now show for  $\mathbb{C}^2$ .

**Theorem 2.14.** *Let  $C$  be a complex curve in  $\mathbb{C}^2$  whose second fundamental form is nonzero at each point. Assume that  $r \in \mathbb{R}_+$  is chosen so that  $C$  has no focal point at distance  $r$ . Then the tube of radius  $r$  around  $C$  is a contact hypersurface of  $\mathbb{C}^2$ .*

*Proof.* Let  $C$  be a complex curve in  $\mathbb{C}^2$  whose second fundamental form is nonzero at each point. Assume that  $r \in \mathbb{R}_+$  is chosen so that  $C$  has no focal point at distance  $r$ . Then the tube  $C_r$  of radius  $r$  around  $C$  is well-defined. Since every complex submanifold of a Kähler manifold is a minimal submanifold, the principal curvatures of  $C$  with respect to a unit normal vector are of the form  $\frac{1}{\theta}$  and  $-\frac{1}{\theta}$  for some  $\theta > 0$ . The corresponding principal curvatures of the tube  $C_r$  at the corresponding point are  $\frac{1}{\theta-r}$  and  $-\frac{1}{\theta+r}$  with the same principal curvature spaces via the usual identification of tangent vectors in a Euclidean space (see Theorem 8.2.2 in [2] for how to calculate the principal curvatures of tubes). These two principal curvature spaces span the maximal complex subspace  $\mathcal{C}$  and hence we get  $S\phi + \phi S = \frac{2r}{\theta^2 - r^2}\phi$ , which shows that  $C_r$  is a contact hypersurface.  $\square$

### 3. CONTACT HYPERSURFACES IN $Q^n$ AND $Q^{n*}$

We now consider the case of the complex quadric  $Q^n = SO_{n+2}/SO_n SO_2$  and its noncompact dual space  $Q^{n*} = SO_{n,2}/SO_n SO_2$ ,  $n \geq 3$ . The complex quadric (and its noncompact dual) have two geometric structures which completely describe its Riemannian curvature tensor  $\bar{R}$ . The first geometric structure is of course the Kähler structure  $(J, g)$ . The second geometric structure is a rank two vector bundle  $\mathcal{A}$  over  $Q^n$  which contains an  $S^1$ -bundle of real structures on the tangent spaces of  $Q^n$ . This bundle has for instance been studied by Smyth in [7] in the context of complex hypersurfaces. The complex quadric  $Q^n$  is a complex hypersurface in  $\mathbb{C}P^{n+1}$  and the bundle  $\mathcal{A}$  is just the family of shape operators with respect to the normal vectors in the rank two normal bundle. We refer also to [3] for more details about  $\mathcal{A}$ . The Riemannian curvature tensor  $\bar{R}$  of  $Q^n$  is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY, \end{aligned}$$

where  $A$  is an arbitrary real structure in  $\mathcal{A}$ . For  $Q^{n*}$  the Riemannian curvature tensor has the same form with a minus sign in front of it. For a real structure  $A \in \mathcal{A}$  we denote by  $V(A)$  its  $(+1)$ -eigenspace; then  $JV(A)$  is the  $(-1)$ -eigenspace of  $A$ . By  $\mathcal{Q}$  we denote the maximal  $\mathcal{A}$ -invariant subbundle of  $TM$ .

A nonzero tangent vector  $W$  of  $Q^n$  resp.  $Q^{n*}$  is called singular if it is tangent to more than one maximal flat in  $Q^n$  resp.  $Q^{n*}$ . There are two types of singular tangent vectors in this situation:

- (i) If there exists a real structure  $A \in \mathcal{A}$  such that  $W \in V(A)$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathcal{A}$ -principal.
- (ii) If there exist a real structure  $A \in \mathcal{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $W/\|W\| = (X + JY)/\sqrt{2}$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathcal{A}$ -isotropic.

For every unit tangent vector  $W$  of  $Q^n$  resp.  $Q^{n*}$  there exist a real structure  $A \in \mathcal{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $W = \cos(t)X + \sin(t)JY$  for some  $t \in [0, \pi/4]$ . The singular tangent vectors correspond to the values  $t = 0$  and  $t = \pi/4$ .

We now apply the results in Section 2 to  $Q^n$  (resp.  $Q^{n*}$ ). Inserting  $X = JN$  and  $Y = Z = N$  into the expression for the curvature tensor  $\bar{R}$  of  $Q^n$ , and using the fact that  $AJ = -JA$ , we get  $\bar{R}(JN, N)N = 4JN + 2g(AN, N)AJN - 2g(AJN, N)AN$ . If  $N$  is  $\mathcal{A}$ -principal, that is,  $AN = N$  for some real structure  $A \in \mathcal{A}$ , then we have  $\bar{R}(JN, N)N = 2JN$ . If  $N$  is not  $\mathcal{A}$ -principal, then there exists a real structure  $A \in \mathcal{A}$  such that  $N = \cos(t)Z_1 + \sin(t)JZ_2$  for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 < t \leq \frac{\pi}{4}$ . This implies  $AN = \cos(t)Z_1 - \sin(t)JZ_2$ ,  $JN = \cos(t)JZ_1 - \sin(t)Z_2$  and  $AJN = -\cos(t)JZ_1 - \sin(t)Z_2$ . Then we have  $g(AN, N) = \cos(2t)$  and  $g(AJN, N) = 0$ , and therefore  $\bar{R}(JN, N)N = 4JN + 2\cos(2t)AJN$ . Thus  $JN$  is an eigenvector of  $\bar{R}_N$  if and only if  $t = \frac{\pi}{4}$  or  $AJN$  is a multiple of  $JN$ . Since both  $\cos(t)$  and  $\sin(t)$  are nonzero for  $0 < t \leq \frac{\pi}{4}$  it is easy to see from the above expressions that  $AJN$  is never a multiple of  $JN$ . Since  $t = \frac{\pi}{4}$  if and only if  $N$  is  $\mathcal{A}$ -isotropic we therefore conclude that  $JN$  is an eigenvector of  $\bar{R}_N$  everywhere if and only if  $N$  is  $\mathcal{A}$ -principal or  $\mathcal{A}$ -isotropic everywhere. The Riemannian curvature tensor of the noncompact dual symmetric space  $Q^{n*}$  is just the negative of the Riemannian curvature tensor of  $Q^n$ . We therefore have proved:

**Proposition 3.1.** *Let  $M$  be a real hypersurface of the complex quadric  $Q^n$  (resp. of its noncompact dual space  $Q^{n*}$ ),  $n \geq 3$ . Then the following statements are equivalent:*

- (i)  $JN$  is an eigenvector of the normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  everywhere;
- (ii)  $N$  is  $\mathcal{A}$ -principal or  $\mathcal{A}$ -isotropic everywhere;
- (iii)  $N$  is a singular tangent vector of  $Q^n$  (resp. of  $Q^{n*}$ ) everywhere.

We now insert  $X = JN$  and  $Y = N$  into the equation for  $\bar{R}$  and assume that  $Z \in TM$ . Then we get  $\bar{R}(JN, N)Z = 2\eta(Z)N + 2JZ + 2g(AN, Z)AJN - 2g(AJN, Z)AN$ . In particular, for  $Z \in \mathcal{C}$  this gives  $\bar{R}(JN, N)JZ = -2Z + 2g(AJN, Z)AJN + 2g(AN, Z)AN$ . If  $Z \in \mathcal{C}$  is a principal curvature vector of  $M$  with corresponding principal curvature  $\lambda$ , we obtain from Proposition 2.6:

**Proposition 3.2.** *Let  $M$  be a contact hypersurface of  $Q^n$  resp. of  $Q^{n*}$ . Then we have*

$$\begin{aligned} & \epsilon(\lambda^2 - 2\rho\lambda + (\alpha\rho + \epsilon))Z \\ &= g(Z, AN)(AN - g(AN, N)N) + g(Z, AJN)(AJN - g(AJN, JN)JN) \end{aligned}$$

for all  $Z \in \mathcal{C}$  with  $SZ = \lambda Z$ , where  $\epsilon = +1$  for  $Q^n$  and  $\epsilon = -1$  for  $Q^{n*}$ .

We will now investigate the normal vector field of a contact hypersurface.

**Proposition 3.3.** *Let  $M$  be a contact hypersurface of  $Q^n$  resp. of  $Q^{n*}$ . Then the normal vector field  $N$  cannot be  $\mathcal{A}$ -isotropic.*

*Proof.* We give the argument for  $Q^n$ , for  $Q^{n*}$  it is analogous. If  $N$  is  $\mathcal{A}$ -isotropic we obtain from Proposition 3.2 that  $(\lambda^2 - 2\rho\lambda + (\alpha\rho + 1))Z = g(Z, AN)AN + g(Z, AJN)AJN = Z_{\mathcal{C} \ominus \mathcal{Q}}$  for all  $Z \in \mathcal{C}$  with  $SZ = \lambda Z$ . We decompose  $Z = Z_{\mathcal{Q}} + Z_{\mathcal{C} \ominus \mathcal{Q}}$  into its  $\mathcal{Q}$ - and  $(\mathcal{C} \ominus \mathcal{Q})$ -components. This implies  $(\lambda^2 - 2\rho\lambda + (\alpha\rho + 1))Z_{\mathcal{Q}} = 0$  and  $(\lambda^2 - 2\rho\lambda + \alpha\rho)Z_{\mathcal{C} \ominus \mathcal{Q}} = 0$ . If  $Z_{\mathcal{C} \ominus \mathcal{Q}} \neq 0$  then  $\lambda^2 - 2\rho\lambda + \alpha\rho = 0$  and therefore  $Z_{\mathcal{Q}} = 0$ . It follows that  $\mathcal{Q}$  and  $\mathcal{C} \ominus \mathcal{Q}$  are invariant under the shape operator of  $M$ . There exists a one-form  $q$  on  $Q^n$  along  $M$  such that  $\bar{\nabla}_X A = q(X)JA$  for all  $X \in TQ^n$  (see e.g. Proposition 7 in [7]). By differentiating the equation  $g(AN, JN) = 0$  with respect to  $X \in TM$  we get  $g(SAJN, X) = 0$ , which implies  $SAJN = 0$ . By differentiating the equation  $g(AN, N) = 0$  and using Proposition 2.3 we get  $0 = g(SAN, X)$  for all  $X \in TM$ , which implies  $SAN = 0$  and thus  $SAJN = 2\rho AJN$ . Altogether this yields  $\rho = 0$ , which is a contradiction. It follows that  $N$  cannot be  $\mathcal{A}$ -isotropic.  $\square$

We now investigate the case when  $N$  is  $\mathcal{A}$ -principal and  $\bar{M} = Q^n$ . If  $N$  is  $\mathcal{A}$ -principal, that is, if  $AN = N$ , we get from Proposition 3.2 that  $(\lambda^2 - 2\rho\lambda + (\alpha\rho + 1))Z = 0$  for all  $Z \in \mathcal{C}$  with  $SZ = \lambda Z$ . Thus there are at most two distinct constant principal curvatures  $\lambda$  and  $\mu = 2\rho - \lambda$  on  $\mathcal{C}$ . We again use the fact that there exists a one-form  $q$  on  $Q^n$  along  $M$  such that  $\bar{\nabla}_X A = q(X)JA$  for all  $X \in TQ^n$ . By differentiating the equation  $g(AN, JN) = 0$  with respect to  $X \in TM$  we get  $q(X) = 2g(ASX, JN) = 2g(SAJN, X) = -2g(SJAN, X) = -2g(SJN, X) = -2\alpha g(JN, X) = 2\alpha\eta(X)$ . It follows that  $\bar{\nabla}_X A = 0$  for all  $X \in \mathcal{C}$ . From  $AN = N$  we get  $AJN = -JAN = -JN$ . Differentiating this equation with respect to  $X \in \mathcal{C}$  gives  $ASX = SX$ . Thus, for all  $Z \in \mathcal{C}$  with  $SZ = \lambda Z$  resp.  $SZ = \mu Z$  we get  $\lambda AZ = \lambda Z$  and  $\mu AZ = \mu Z$ . If both  $\lambda$  and  $\mu$  are nonzero this implies  $AZ = Z$  for all  $Z \in \mathcal{C}$  and hence  $\text{tr}(A) = 2(n - 1)$ , which contradicts the fact that  $A$  is a real structure and hence  $\text{tr}(A) = 0$ . We thus may assume that  $\lambda = 0$ . If the corresponding principal curvature space  $T_\lambda$  is  $J$ -invariant this implies  $\rho = 0$ , which is a contradiction. We thus must have  $0 \neq \mu = 2\rho$  and  $JT_\lambda = T_\mu$ . Thus we have shown that there are exactly two distinct constant principal curvatures  $\lambda = 0$  and  $\mu = 2\rho$  on  $\mathcal{C}$ . Moreover, we have  $JT_\lambda = T_\mu$  for the corresponding principal curvature spaces  $T_\lambda$  and  $T_\mu$ . From Proposition 3.2 we also get the equation  $\alpha\rho + 1 = 0$ .

For the dual manifold  $Q^{n*}$  we have to consider the equation  $(\lambda^2 - 2\rho\lambda + (\alpha\rho - 1))Z = 0$ , but all other arguments remain the same. Thus we have proved:

**Proposition 3.4.** *Let  $M$  be a contact hypersurface of  $Q^n$  resp. of  $Q^{n*}$  and assume that the normal vector field  $N$  is  $\mathcal{A}$ -principal. Then  $M$  has three distinct constant principal curvature  $\alpha$ ,  $\lambda = 0$  and  $\mu = 2\rho$  with corresponding principal curvature spaces  $\mathbb{R}JN$ ,  $T_\lambda \subset \mathcal{C}$  and  $T_\mu \subset \mathcal{C}$  satisfying  $JT_\lambda = T_\mu$ . The principal curvature  $\alpha$  is given by  $\alpha = -\frac{1}{\rho}$  when  $\bar{M} = Q^n$  and  $\alpha = \frac{1}{\rho}$  when  $\bar{M} = Q^{n*}$ . Moreover,  $T_\mu = V(A) \cap \mathcal{C}$  and  $T_\lambda = JV(A) \cap \mathcal{C}$ .*

We now prove Theorem 1.1. Without loss of generality we may assume that  $\rho > 0$  (otherwise replace  $N$  by  $-N$ ). Since  $\rho$  is constant there exists  $0 < r < \frac{\pi}{2\sqrt{2}}$  such that  $\rho = \frac{1}{\sqrt{2}} \tan(\sqrt{2}r)$ . From Proposition 3.4 we then get  $\alpha = -\sqrt{2} \cot(\sqrt{2}r)$  and  $\mu = \sqrt{2} \tan(\sqrt{2}r)$ .

The normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  has two eigenvalues 0 and 2 with corresponding eigenspaces  $T_\lambda \oplus \mathbb{R}N$  and  $T_\mu \oplus \mathbb{R}JN$ . Thus the normal Jacobi operator and the shape operator of  $M$  commute, which allows us to use Jacobi field theory to determine explicitly the focal points of  $M$ . For  $p \in M$  we denote by  $\gamma_p$  the geodesic in  $Q^n$  with  $\gamma_p(0) = p$  and  $\dot{\gamma}_p(0) = N_p$ , and by  $F$  the smooth map  $F : M \rightarrow Q^n$ ,  $p \mapsto \gamma_p(r)$ . The differential  $d_p F$  of  $F$  at  $p$  can be computed using Jacobi vector fields by means of  $d_p F(X) = Y_X(r)$ , where

$Y_X$  is the Jacobi vector field along  $\gamma_p$  with initial values  $Y_X(0) = X$  and  $Y'_X(0) = -SX$ . This leads to the following expressions for the Jacobi vector fields along  $\gamma_p$ :

$$Y_X(r) = \begin{cases} (\cos(\sqrt{2}r) - \frac{\alpha}{\sqrt{2}} \sin(\sqrt{2}r))E_X(r) & \text{if } X \in T_\alpha, \\ (\cos(\sqrt{2}r) - \frac{\mu}{\sqrt{2}} \sin(\sqrt{2}r))E_X(r) & \text{if } X \in T_\mu, \\ E_X(r) & \text{if } X \in T_\lambda, \end{cases}$$

where  $E_X$  is the parallel vector field along  $\gamma_p$  with  $E_X(0) = X$ . This shows that  $\text{Ker}(dF) = T_\mu$  and thus  $F$  has constant rank  $n$ . Therefore, locally,  $F$  is a submersion onto a submanifold  $P$  of  $Q^n$  of real dimension  $n$ . Moreover, the tangent space  $T_{F(p)}P$  of  $P$  at  $F(p)$  is obtained by parallel translation of  $(T_\lambda \oplus T_\alpha)(p)$  along  $\gamma_p$ . Thus the submanifold  $P$  is a totally real submanifold of  $Q^n$  of real dimension  $n$ .

The vector  $\eta_p = \dot{\gamma}_p(r)$  is a unit normal vector of  $P$  at  $F(p)$  and the shape operator  $S_{\eta_p}$  of  $P$  with respect to  $\eta_p$  can be calculated from the equation  $S_{\eta_p}Y_X(r) = -Y'_X(r)$ , where  $X \in (T_\lambda \oplus T_\alpha)(p)$ . The above expression for the Jacobi vector fields  $Y_X$  implies  $Y'_X(r) = 0$  for  $X \in T_\lambda(p)$  and  $X \in T_\alpha(p)$ , and therefore  $S_{\eta_p} = 0$ . The vectors of the form  $\eta_q$ ,  $q \in F^{-1}(\{F(p)\})$ , form an open subset of the unit sphere in the normal space of  $P$  at  $F(p)$ . Since  $S_{\eta_q}$  vanishes for all  $\eta_q$  it follows that  $P$  is an  $n$ -dimensional totally geodesic totally real submanifold of  $Q^n$ . Rigidity of totally geodesic submanifolds now implies that the entire submanifold  $M$  is an open part of a tube of radius  $r$  around an  $n$ -dimensional connected, complete, totally geodesic, totally real submanifold  $P$  of  $Q^n$ . Such a submanifold is also known as a real form of  $Q^n$ . The real forms of the complex quadric  $Q^n$  are well-known, see for example [4] or [5]. This shows that  $P$  is either a sphere  $S^n$  or  $(S^a \times S^b)/\{\pm I\}$  with  $a+b = n$  and  $a, b \geq 1$ . However, we see from the above calculations that at each point the tangent space of  $P$  corresponds to the  $(-1)$ -eigenspace of a real structure on  $Q^n$ , which rules out  $(S^a \times S^b)/\{\pm I\}$ . It follows that  $P$  is a sphere  $S^n$  embedded in  $Q^n$  as a real form.

We remark that the focal set of a real form  $S^n$  in  $Q^n$  is a totally geodesic complex hyperquadric  $Q^{n-1} \subset Q^n$ . So the tubes around  $S^n$  can also be regarded as tubes around  $Q^{n-1}$ .

We now proceed with the proof of Theorem 1.2. We again assume  $\rho > 0$ . We can write  $\rho = \frac{1}{\sqrt{2}} \tanh(\sqrt{2}r)$ ,  $\rho = \frac{1}{\sqrt{2}}$  or  $\rho = \frac{1}{\sqrt{2}} \coth(\sqrt{2}r)$  with some  $r \in \mathbb{R}_+$ . The normal Jacobi operator  $\bar{R}_N = \bar{R}(\cdot, N)N$  has two eigenvalues 0 and  $-2$  with corresponding eigenspaces  $T_\lambda \oplus \mathbb{R}N$  and  $T_\mu \oplus \mathbb{R}JN$ . The Jacobi vector fields are given by

$$Y_X(r) = \begin{cases} (\cosh(\sqrt{2}r) - \frac{\alpha}{\sqrt{2}} \sinh(\sqrt{2}r))E_X(r) & \text{if } X \in T_\alpha, \\ (\cosh(\sqrt{2}r) - \frac{\mu}{\sqrt{2}} \sinh(\sqrt{2}r))E_X(r) & \text{if } X \in T_\mu, \\ E_X(r) & \text{if } X \in T_\lambda, \end{cases}$$

We now distinguish three cases:

Case 1:  $\rho = \frac{1}{\sqrt{2}} \tanh(\sqrt{2}r)$ . From Proposition 3.4 we then get  $\alpha = \sqrt{2} \coth(\sqrt{2}r)$  and  $\mu = \sqrt{2} \tanh(\sqrt{2}r)$ . Here we get  $\text{Ker}(dF) = T_\alpha$  and thus  $F$  has constant rank  $2(n-1)$ . Analogously to the compact case we can deduce that  $M$  is an open part of a tube around a totally geodesic, complex submanifold  $P$  of  $Q^{n*}$  of complex dimension  $n-1$ . Using duality of symmetric spaces and [5] we can see that  $P$  is a totally geodesic  $Q^{(n-1)*}$  in  $Q^{n*}$ .

Case 2:  $\rho = \frac{1}{\sqrt{2}}$ . From Proposition 3.4 we then get  $\alpha = \mu = \sqrt{2}$ . In this case  $M$  does not have any focal points. For  $X \in T_\alpha$  or  $X \in T_\mu$  the Jacobi vector fields are

$Y_X(r) = \exp(-\sqrt{2}r)E_X(r)$  and remain bounded for  $r \rightarrow \infty$ . Together with  $Y_X(r) = E_X(r)$  for  $X \in T_\lambda$  this implies that all normal geodesics of  $M$  are asymptotic to each other. From this we see that  $M$  is a horosphere whose center at infinity is given by an equivalence class of asymptotic geodesics whose tangent vectors are all  $\mathcal{A}$ -principal. Thus the center at infinity of the horosphere is a singular point of type  $\mathcal{A}$ -principal.

Case 3:  $\rho = \frac{1}{\sqrt{2}} \coth(\sqrt{2}r)$ . From Proposition 3.4 we then get  $\alpha = \sqrt{2} \tanh(\sqrt{2}r)$  and  $\mu = \sqrt{2} \coth(\sqrt{2}r)$ . In this situation we get  $\text{Ker}(dF) = T_\mu$  and thus  $F$  has constant rank  $n$ . This case is analogous to the compact situation and we deduce that  $M$  is an open part of a tube around a real form  $\mathbb{R}H^n$  of  $Q^{n*}$ .

The arguments given above can be carried out in reverse order to show that all the resulting hypersurfaces are in fact contact hypersurfaces. This finishes the proof of Theorems 1.1 and 1.2

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